Received: *June 19, 2001*, Accepted: *June 27, 2001*

On non-abelian structures in field theory of open strings

Anton A. Gerasimov

Institute for Theoretical and Experimental Physics
Moscow, 117259, Russia
E-mail: gerasim@vxitep.itep.ru

Samson L. Shatashvili*

Department of Physics, Yale University
New Haven, CT 06520-8120, USA
E-mail: samson.shatashvili@yale.edu

Abstract: Multi-brane backgrounds are studied in the framework of the background independent open string field theory. A simple description of the non-abelian degrees of freedom is given. Algebra of the differential operators acting on the space of functions on the space-time provides a natural tool for the discussion of this phenomena.

Keywords: Bosonic Strings, D-branes, Conformal Field Models in String Theory, Brane Dynamics in Gauge Theories.

*On leave of absence from St. Petersburg Branch of Steklov Mathematical Institute, Fontanka, St. Petersburg, Russia.

JHEP06(2001)066

Contents

1. Introduction **1**

2. Matrix algebra, vertex operators and $\mathrm{GL}(\infty)$ **3**

3. Quantum analog of the hamiltonian reduction on T^*X **and matrix algebra** **8**

4. Algebra of the differential operators from the boundary field theory **9**

5. Conclusions and further directions **12**

1. Introduction

The understanding of the relation between (large N) gauge theories and string theories [1]–[4] (see also discussion in [5]), is probably one of the most important problems in the modern theoretical physics. During the last decade this connection showed up in string theory literature with various faces: multiple D-branes backgrounds [6], Matrix Theory proposal [7], AdS/CFT correspondence [8, 9, 10], various constructions of the solitonic objects in terms of matrix degrees of freedom [11]–[14]. All this implies very interesting interrelation between the non-abelian gauge degrees of freedom (of open strings as the closest relatives of gauge theories in the string world) and purely geometrical "gravitational" degrees of freedom (of closed strings). From the mathematical side it seems to provide the most intriguing example of the unity of Algebra and Geometry.

The simplest example of such relation is given by multiple D-brane background. The important lesson learned from the recent studies of the open string field theory is the new point of view on such backgrounds. A few years ago one would think about D-branes as solitons in the closed string theory, but by now it is well known that D-branes can be also considered as solitons in the open string field theory. This can be seen both from cubic **CS** string field theory of [15] and background independent open string field theory [16]–[19]. In the latter any boundary CFT (with world-sheet being a disk) is a critical point for corresponding string field theory action; this action evaluated on classical solution coincides with the world-sheet partition function. Thus any D-brane background should provide a classical solution for the background independent open string field theory lagrangian.[1]

Although the above argument gives simple explanation of the fact that D-branes are solitons in open string field theory, many issues remain unclarified. For instance it is not clear how the non-abelian gauge fields emerge in the case of the near coincident D-branes in this formalism. The interpretation of the closed strings in these terms also remains an open question (see discussion in lecture notes [21] and comments below).

In this paper we will address the first question using the formalism of background independent open string field theory which turned out to be very effective in verifying the Sen's conjectures [22] as it was demonstrated in [23, 24, 25].[2]

One of the main reasons for the appearance of the non-abelian algebraic structures in the open string theory is an obvious geometrical interpretation of the open string as a some kind of matrix (open strings have two ends) acting in the appropriate Hilbert space [28]–[32]. The infinite dimensionality of this Hilbert space leads to some unusual phenomena responsible for the peculiar properties of string theory. Let us stress that the non-commutativity is a basic property of string theory and is not a special feature of the backgrounds with non-zero vacuum value of the B-field.

The existence of this "matrix" structure of the open string theory allows us to add non-abelian indexes to the open string wave function with the help of Chan-Paton factors. Basically one has a tensor product of the stringy "matrix" algebra and some abstract matrix algebra. One of the important outputs of the discovery of the D-branes was the understanding that Chan-Paton factors also have a geometrical origin and naturally appear in multiple D-branes backgrounds [6].

We argue that the non-commutative structure of the open string theory on a manifold M may be captured in a reasonable approximation by the non-commutative algebra of differential operators on M. The algebra of differential operators on the manifold together with its various completions is the model of the matrix algebra appearing in Matrix Theory of strings [7]. By considering the algebra of differential operators instead of the (abstract) infinite dimensional matrices of the Matrix Theory we partially loose background independence but instead we get a very explicit description of the non-commutative degrees of freedom.

[1]In fact this was already noticed in [17] and reinterpreted in modern terms in [20] in order to argue that field theory of open strings contains all possible stringy backgrounds, including D-branes and closed strings, as classical solutions and thus can serve as a definition of full (consistent) off-shell string theory.

[2]After this work was finished the studies of similar questions in the lines of cubic **CS** string field theory appeared in [26, 27].

The algebra of differential operators may be considered as a non-commutative deformation of the algebra of functions on the cotangent bundle T^*M defined by the canonical quantization with the symplectic form $\omega = \sum_i dp_i \wedge dx^i$. The widely discussed non-commutative structure appearing when B-field is non zero [33, 34, 35], from this point of view, may be described as some B-dependent deformation of the basic symplectic structure. Full description of the non-commutative structure in the open string theory obviously includes the differential operators on the space of the maps of the (holomorphic) disks to the space-time. Some remarks on this are given at the end of the paper.

We propose a construction of the vertex operators responsible for Chan-Paton degrees of freedom. The main ingredients of this construction turns out to be closely related to the algebra of differential operators. It is not accidental. By looking at the short distance correlation functions in the boundary conformal field theory with appropriate regularization prescription we show that they can be reproduced by a simple quantum mechanics describing the quantization of the space which locally looks like a cotangent bundle of the space-time where string theory is defined. The corresponding algebra of the quantum operators (differential operators acting on the space of functions on space-time) is the basic structure one needs in order to extract the matrix degrees of freedom entering in the description of the D-brane theories. We give the explicit construction of this reduction in terms of tachyon condensation for various tachyonic profiles.

We hope that our simple model for the description of the non-abelian degrees of freedom of the open string theory, among other things could shed some light on the open/closed strings correspondence in the lines of discussion in [25]. As an illustration of the usefulness of the approach proposed in this paper, we apply it [36] to the problem of the universal description of the RR gauge field couplings with open strings.

One shall note that remarks regarding the importance of the understanding of non-abelian degrees of freedom in terms of boundary sigma models can be found in [24, 37]. The considerations in [38] might also be relevant to our discussion.

2. Matrix algebra, vertex operators and $\mathrm{GL}(\infty)$

In background independent open string field theory (we will consider only the case when ghosts and matter decouple) the classical action is defined on the space of boundary conditions for bosonic string with world-sheet of disk topology. This action has a critical point for any conformal boundary condition; also, every critical point corresponds to a conformal boundary condition. If the space of boundary conditions is parameterized by local coordinates t^i the classical action can be written in terms of the β-function $\beta^i(t)$ corresponding to the coupling t^i and matter partition function $Z(t)$:

$$S(t) = -\beta^i(t)\partial_i Z(t) + Z(t)\,. \tag{2.1}$$

On-shell ($t = t_c$ — exactly marginal boundary perturbation) the action coincides with the partition function $Z(t_c)$. If the world-sheet theory is perturbed by the complete set of boundary operators (corresponding to the full spectrum of given conformal field theory) the action (2.1) is well defined in the space of couplings although at first glance the world-sheet theory might look non renormalizable [19]. The simplest case where one can exactly compute the effective action is given by the quadratic off-shell tachyon profile [17]:

$$T(X(\theta)) = T_0 + \sum_\mu \frac{1}{2}u_\mu(X^\mu(\theta) - a^\mu)^2 \tag{2.2}$$

(for the discussion of the limitations of this computation for current application see [21]).

Since various collection of D-branes shall provide us with world-sheet (disk) conformal field theories, the action (2.1) must have corresponding critical points. We shall show that multi-branes are indeed the solutions of the equations of motion for (2.1). Besides, there shall be a vacuum without open strings [22] — closed string vacuum. In terms of the profile, (2.2) the closed string vacuum corresponds to the solution with $T_0 = \infty$, $u_\mu = 0$ of the equations of motion for S. Solutions corresponding to D-branes can be represented by the tachyonic profiles localizing the effective action on the relevant submanifolds. For example, in order to explicitly write down such solution one could use the approximation of the delta function by the profile (2.2) with $T_0 \to \infty$, $u_1 \to \infty$, $u_{\mu\neq 1} = 0$. This corresponds to D-brane boundary condition which is described by the equation $X^1 = a^1$. Let us consider decomposition of the fields on the boundary of the disk $X^\mu(\theta) = X^\mu + X^\mu_*(\theta)$ on the constant and non-constant parts, integrate out the non-constant modes and find the corresponding action in terms of the integral over X (here we use integration by parts over X; for some details see [21]):

$$\begin{aligned} S(T_0, u) &= \int e^{-T(X)} F(\partial^2 T(X), \partial^3 T(X), \ldots) \\ &= \int dX e^{-\sum_\mu \frac{1}{2}u_\mu(X^\mu - a^\mu)^2} F(T_0, u)\,. \end{aligned} \tag{2.3}$$

One can solve the equations of motions for T_0 and evaluate the action on such solution:

$$S(u) = \int dX e^{-\sum_\mu \frac{1}{2}u_\mu(X^\mu - a^\mu)^2} \widetilde{F}(u)\,. \tag{2.4}$$

Consider the case when $u_{\mu\neq 1} = 0$. The action at the conformal point should be invariant with respect to the scaling of u_1. This unambiguously fixes the asymptotics

JHEP06(2001)066

of function $\widetilde{F}(u_1)$:

$$S(u_1) \sim \int dX e^{-\frac{1}{2}u_1(X^1-a^1)^2}\sqrt{u_1}\,. \tag{2.5}$$

In the limit $u_1 \to \infty$ we have:

$$S(u) \longrightarrow \int dX \delta(X^1 - a^1)\,. \tag{2.6}$$

Note the correct normalization of the action is a consequence of the fact that at the conformal point the action coincides with the partition function. Thus we see the localization which is a manifestation of the appearance of the D-brane. In terms of the sigma model description this leads to the projector operator:

$$\begin{aligned}\int DX e^{\int d\theta(-uX^2(\theta))} \sim \int DXDP e^{\int d\theta\left(\frac{1}{u}P^2(\theta)+P(\theta)(X(\theta)-a)\right)} \longrightarrow_{u\to\infty} \times \\ \times \int DX \delta(X-a)\,. \end{aligned} \tag{2.7}$$

For general tachyon potential an arbitrary map of the disk to the flat space-time can be written as a sum of harmonic map (with boundary map being $X(\theta)$) and arbitrary function with zero value on the boundary. Thus the world-sheet action perturbed by the boundary operators (as in background independent open string field theory) can be written purely in terms of the boundary value $X(\theta)$:

$$\begin{aligned} I_B &= \int\int d\theta d\theta' X^\mu(\theta) H(\theta,\theta') X_\mu(\theta') + \int d\theta T(X(\theta)) \\ &= T(X) + \int X_*^\mu(\theta)[H(\theta-\theta')\delta_{\mu\nu} + \delta(\theta-\theta')\partial_\mu\partial_\nu T(X)]X_*^\nu(\theta') + \\ &\quad + O(\partial^3 T(X)) \end{aligned} \tag{2.8}$$

here $H(\theta,\theta') = \frac{1}{2}\sum_{k\in\mathbb{Z}} e^{ik(\theta-\theta')}|k|$. In the lowest, two derivative approximation one can ignore the last term and get for the world-sheet partition function:

$$Z(T) = \int dX e^{-T(X)} \det{}'[H + \partial^2 T]^{-1/2}\,. \tag{2.9}$$

This can be used for the computation of (2.1) (but only in two derivative approximation as it was mentioned in [21]; for higher derivative terms the contributions of the last term in (2.8) mix with the powers of $\partial^2 T$ coming from determinant in (2.9)).

Now consider the case of the multicritical tachyon profile, or in other words — some kind of boundary Landau-Ginzburg theory with potential $W(X)^2$ (for simplicity we will be treating only the case of non-trivial dependence on the one coordinate of space-time — X):

$$T(X(\theta)) = T_0 + u(W(X))^2 = T_0 + u\prod_{i=1}^{N}(X-a_i)^2\,. \tag{2.10}$$

At $u \to \infty$ (when the points a_i are far from each other) in this case we get the insertion of the projector in the world-sheet path integral which defines the space-time action (2.1) (from the superposition of the solutions from different critical points of the tachyon profile) and thus:

$$S(u) \sim \sum_i \int dX e^{-\frac{1}{2}uT''(a_i)(X-a_i)^2} \sqrt{uT''(a_i)} \longrightarrow \int dX \left(\sum_i \delta(X-a_i) \right) . \quad (2.11)$$

Note that the normalization in front of gaussian is uniquely fixed by considerations described above.

Thus, world-sheet path integral (and the string field theory action) in the boundary theory reduces to the trace over n-dimensional vector space as a result of above projection. This vector space may described as a subspace (linear combination of the delta-functions) of the linear space $\mathcal{F}$ of all functions on the space-time.

Our goal will be to describe the operators corresponding to "jumps" between the points of minima of the potential $T(X)$.

First we give a simple description in terms of the operators acting on the space $\mathcal{F}$. The operators we need should be compatible with the reduction (2.11) (they should preserve the linear space of the delta-functions $\delta(X-a_i)$). Let us first note that there are projection operators:

$$E_{ii} = \frac{\prod_{k\neq i}(X-a_k)}{\prod_{k\neq i}(a_i-a_k)} = \frac{W(X)}{W'(a_i)(X-a_i)} \quad (2.12)$$

with the property:

$$\begin{aligned} E_{ii}(X)\delta(X-a_j) &= \delta_{ij}\delta(X-a_j)\,, \\ E_{ii}(X)E_{jj}(X) &= \delta_{ij}E_{ii}(X) \mod W(X)\,. \end{aligned} \quad (2.13)$$

In addition one can consider the set of differential (more exactly difference) operators:

$$E_{ij} = e^{(a_i-a_j)\partial_X} \frac{\prod_{k\neq j}(X-a_k)}{\prod_{k\neq j}(a_j-a_k)} = e^{(a_i-a_j)\partial_X} E_{jj} \quad (2.14)$$

(in the special case $i=j$, E_{ij} becomes E_{ii} defined in (2.12)). Their action does not spoil the structure defined by reduction (2.11). It is easy to show that the following relation is satisfied:

$$E_{ij}\delta(X-a_k) = \delta_{jk}\delta(X-a_i)\,. \quad (2.15)$$

We conclude that the operators E_{ij} modulo $W(X)$ generate the algebra gl(N).

Now we would like to give an explicit realization of these operators in terms of the operators of the corresponding boundary conformal field theory. The non-abelian degrees of freedom have their origin in the open strings stretching between D-branes. Let us start with Dirichlet boundary condition $X_{|\partial D} = a$. It is easy to construct the boundary operator in conformal field theory that shifts the Dirichlet boundary conditions $X_{|\partial D} = a$ to new boundary condition $X_{|\partial D} = a + \Delta X(\theta)$:

$$V_{\Delta X} = e^{\int d\theta \partial_n X(\theta) \Delta X(\theta)} . \tag{2.16}$$

In order to realize the process when in the beginning the string ends on the D-brane $X = a_i$ then jumps to D-brane $X = a_j$ and then at the end jumps back to the D-brane $X = a_i$ we take the "shift"-function in the form of step-function:

$$\begin{aligned} \Delta X &= (a_1 - a_2)\epsilon(\theta|\theta_1, \theta_2) , \\ \epsilon(\theta|\theta_1, \theta_2) &= 1 , \qquad \theta_1 < \theta < \theta_2 , \\ \epsilon(\theta|\theta_1, \theta_2) &= 0 , \qquad \text{otherwise} . \end{aligned} \tag{2.17}$$

Then using the decomposition of the scalar field $X = X_+ + X_-$ in terms of positive and negative frequency modes

$$X^\mu(\theta) = X^\mu_+(\theta) + X^\mu_-(\theta) = \left(\frac{1}{2} X^\mu_0 + \sum_{k>0} X^\mu_k e^{ik\theta} \right) + \left(\frac{1}{2} X^\mu_0 + \sum_{k<0} X^\mu_k e^{ik\theta} \right) \tag{2.18}$$

we could represent $V_{\Delta X}$ operator as the product of two "jump"-operators:

$$\begin{aligned} V_{\Delta X} &= e^{\int_{\theta_2}^{\theta_1} (a_i - a_j) \partial_n X} = e^{\int_{\theta_2}^{\theta_1} (\partial_\theta X_+ - \partial_\theta X_-)(a_i - a_j)} \\ &= e^{(X_+(\theta_1) - X_-(\theta_1))(a_i - a_j)} e^{-(X_+(\theta_2) - X_-(\theta_2))(a_i - a_j)} . \end{aligned} \tag{2.19}$$

Therefore one has the following expression for the "jump"-operator:

$$V^{ij}(X(\theta)) = e^{\frac{(a_i - a_j)}{\alpha'} (X_m(\theta))} , \qquad X_m = X_+ - X_- . \tag{2.20}$$

This operator corresponds to the excitation with the mass defined by its conformal dimension:

$$m^2 = \frac{|a_i - a_j|^2}{(\alpha')^2} \tag{2.21}$$

which is in accordance with the expectations ($m^2 \sim \text{length}^2$) for the mass of the non-abelian part of the tachyon. In order to get the operators describing the general states from the spectrum one should multiply (2.20) by the polynomial of the derivatives of X and the standard exponential factors e^{ipx} responsible for the non-zero momentum along D-brane. One can make a contact with (2.14) by realizing that the sector of the boundary theory corresponding to the Dirichlet boundary condition $X = a_j$ is given by the insertion of the $\delta(X - a_j)$. The action of E_{ij} on this function reproduces the action of the operator $\exp((a_i - a_j)\partial_X)$.

3. Quantum analog of the hamiltonian reduction on T^*X and matrix algebra

In this section we explain the meaning of the operators E_{ij} in a slightly different way. The following can be formulated as the quantum reduction of the algebra of quantized functions on T^*X [39] (this construction also appears in the discussion of W-geometry [40]).

Consider the simple case of one-dimensional flat space M. The differential operator may be considered as the function on the non-commutative space T^*M_h — the deformation of the cotangent bundle defined by the canonical Poisson structure $h\frac{\partial^2}{\partial x \partial p}$. Let $D = \partial^n$ be a differential operator of n-th order. The space of solutions is $\mathrm{Sol}(D) = \{1, x, x^2, \ldots, x^{n-1}\}$.

We would like to make quantum analog of the hamiltonian reduction with the momentum given by D. In other words we look at the subspace $\mathcal{A}$ of all differential operators with the following properties. For any operator $P \in \mathcal{A}$ there should exist an operator P' satisfying the condition:

$$DP = P'D \tag{3.1}$$

and we identify the operators in $\mathcal{A}$ under the equivalence relation:

$$P \sim P + P''D\,, \tag{3.2}$$

where P'' is an arbitrary operator. It is easy to see that the operators from $\mathcal{A}$ act on the space of solutions $\mathrm{Sol}(D)$. Moreover we have a multiplication structure on $\mathcal{A}$ (the product of equivalence classes is unambiguously defined as the equivalence class). Using the second condition we may restrict ourselves by the operators of the order $(n-1)$: $P = a_0 + a_1\partial + \cdots + a_{n-1}\partial^{n-1}$. From the first condition we get the restrictions on the coefficients a_i. Simple calculation shows that the number of free parameters is n^2. It is not surprising that the n^2-dimensional algebra acting on the n-dimensional space is isomorphic to $\mathrm{gl}(n)$. For example, in case of $n = 2$ we have explicit isomorphism:

$$\begin{aligned} P &= (p_0 + p_1 x) + (p_3 + p_2 x - p_1 x^2)\partial\,, \\ P &\sim \begin{pmatrix} p_0 & p_1 \\ p_3 & (p_0 + p_2) \end{pmatrix}\,. \end{aligned} \tag{3.3}$$

This is a standard representation of the $\mathrm{gl}(2)$ algebra in terms of the first order operators.

Consider more interesting example: $D = \partial^2 - \lambda\partial$. The space of solutions $\mathrm{Sol}(D) = (1, e^{\lambda x})$ and the corresponding algebra is generated by the operators:

$$P = (p_1 e^{\lambda x} + p_0) + \left(-\frac{p_1}{\lambda}e^{\lambda x} + \frac{p_2}{\lambda}e^{-\lambda x} + \frac{(p_3 - p_0)}{\lambda}\right)\partial_x\,. \tag{3.4}$$

One could make Fourier transformation and look at the ideal defined by the function $W(x) = x^2 - \lambda x$. The space of solutions $f \in \mathrm{Sol}(W)$ of the equation $W(X)f(X) = 0$ is given by delta-functions $\mathrm{Sol}(W(x)) = \{\delta(x), \delta(x-\lambda)\}$ and using the Fourier transform of (3.4) we get:

$$P = \frac{1}{\lambda}\left(-p_1 e^{-\lambda\partial_x}(x-\lambda) + p_2 e^{\lambda\partial_x} x + p_3 - p_0(x-\lambda)\right). \tag{3.5}$$

The action of these operators on the delta-function bases of $\mathrm{Sol}(W)$ provides the connection with (2.20). These considerations could be straightforwardly generalized to the case of an arbitrary (multidimensional) potential $W(x)$.

4. Algebra of the differential operators from the boundary field theory

In the last section we will show how the framework of the qunatization of the cotangent bundle of the space-time naturally appears in the sigma model description of open strings.

As a model example we consider bosonic sigma-model on the disk with unspecified boundary conditions:

$$S_{2D} = \int d\theta d\tau ||dX||^2 . \tag{4.1}$$

The action takes extreme values on the configurations:

$$\begin{aligned} \Delta X &= 0\,, \\ \partial_n X|_{\partial D} &= 0\,, \end{aligned} \tag{4.2}$$

where ∂_n is the normal derivative on the boundary. Thus on the classical solutions Neumann boundary conditions hold. Boundary action was given in (2.8):

$$I_B = \int\int d\theta d\theta' X_\mu(\theta) H(\theta,\theta') X^\mu(\theta') . \tag{4.3}$$

Here $H = d*$ is a normal derivative operator acting on the harmonic functions on the disk. The meaning of this notation is the following. Introduce the operation $*$ on the fields ("T-duality"):

$$*X^\mu(\theta) = \sum_{k\in\mathbb{Z}} \mathrm{sign}(k)\, X^\mu_k e^{ik\theta} = X^\mu_+(\theta) - X^\mu_-(\theta) \tag{4.4}$$

(compare with (2.18)) $d*$ may be considered as the composition of the usual derivative and $*$-operator. Note that at least when the local operators are considered the fields

$*X$ and X may be treated as independent. Two-point correlation functions of the fields are given by:

$$\begin{aligned}
\langle X^\mu(\theta)X^\nu(\theta')\rangle &= \delta^{\mu\nu}\sum_{k\neq 0}\frac{1}{|k|}e^{ik(\theta-\theta')}\,,\\
\langle *X^\mu(\theta) * X^\nu(\theta')\rangle &= \delta^{\mu\nu}\sum_{k\neq 0}\frac{1}{|k|}e^{ik(\theta-\theta')}\,,\\
\langle X^\mu(\theta) * X^\nu(\theta')\rangle &= \delta^{\mu\nu}\sum_{k\neq 0}\frac{1}{k}e^{ik(\theta-\theta')}\,.
\end{aligned} \tag{4.5}$$

The non-symmetric part of these correlators leads to non-commutativity of the operators $*X$ and X:

$$\begin{aligned}
[X^\mu(\theta), X^\nu(\theta')] &= 0\,,\\
[*X^\mu(\theta), *X^\nu(\theta')] &= 0\,,\\
[*X^\mu(\theta), X^\nu(\theta')] &\sim \delta^{\mu\nu}\delta(\theta-\theta')\,.
\end{aligned} \tag{4.6}$$

The symmetric part of the correlation functions is given by the standard Green function:

$$G(\theta,\theta') = 2\log|e^{i\theta}-e^{i\theta'}|^2 \tag{4.7}$$

and is singular when the points coincide. This leads to the necessity to regularize the theory. Let us suppose that our regularization prescription modifies the metric on the boundary as follows:

$$\begin{aligned}
||z_1-z_2|| &= |z_1-z_2|\,, \qquad \text{when } |z_1-z_2| \gg l_R\,,\\
||z_1-z_2|| &= 1\,, \qquad \text{when } |z_1-z_2| \ll l_R\,,
\end{aligned} \tag{4.8}$$

(thus l_R is a characteristic regularization length). With this prescription we get for modified Green function $G_R(0)=0$. Note that this regularization does not modify the "non-commutative" part of the correlators. Thus we would like to conclude that if we are interested in the small distance correlation functions only "non-commutative" parts survive. One should propose that this is the way matrix degrees of freedom show up in the sigma model approach. It would be interesting to extract relevant degrees of freedom explicitly. This can be done as follows. Let us start with the action:

$$I_B = \int d\theta X^\mu d * X^\mu\,. \tag{4.9}$$

It is obvious that this theory is equivalent to the following theory with the additional fields X^m_μ:

$$I'_B = \int d\theta(X^m_\mu dX^\mu + X^m_\mu d * X^m_\mu)\,. \tag{4.10}$$

To have equivalence between the theories (4.9) and (4.10) one should not integrate over zero modes of the new fields X^m_μ in the functional integral (it may be achieved by insertion of the delta function $\delta(X^m_\mu(0))$).

Now consider a simple example of the calculation of the correlation function in this theory:

$$\left\langle \prod_i e^{ip_i X(\theta_i)} \right\rangle_{I'_B} . \tag{4.11}$$

It is clear that when the points θ_i are sufficiently close correlation functions at small distances are dominated by the first term in the lagrangian (4.10) (with the regularization (4.8)). Thus we are getting the effective theory:

$$I = \int d\theta (X^m_\mu dX^\mu) . \tag{4.12}$$

This is the standard description of the quantization of the cotangent bundle. The basic operators have the commutation relations:

$$\begin{aligned} [X^m_\mu, X^m_\nu] &= 0 , \\ [X^\mu, X^\nu] &= 0 , \\ [X^m_\mu, X^\nu] &= \delta^\nu_\mu . \end{aligned} \tag{4.13}$$

Let us show that the correlation functions of the operators (2.19) and (2.20) reduce to the standard representation of the differential operators in the theory with the action (4.10). Consider the correlation function that includes normal derivatives:

$$\left\langle \prod_i e^{ip_i X(\theta_i)} e^{\int d\theta a_\mu(\theta) d{*}X^\mu(\theta)} \right\rangle_{I'_B} . \tag{4.14}$$

One could get rid of the non-local terms by the shift of the variable. For the correlation function (4.14) the simple shift:

$$X^m_\mu \longrightarrow X^m_\mu + {*}a_\mu \tag{4.15}$$

makes the necessary change. We are interested in the short distance behaviour of the correlation functions. Thus after dropping the terms with non localities with * operations (this is legitimate with our regularization prescription), we come to the following correlation function in the theory with the action (4.10):

$$\left\langle \prod_i e^{ip_i X(\theta_i)} e^{\int d\theta a^\mu(\theta) dX^m_\mu(\theta)} \right\rangle_{I'_B} . \tag{4.16}$$

Now let us take the function a of the form: $a(\theta) = (a_1 - a_2)\epsilon(\theta|\theta_1, \theta_2)$ (compare with (2.17)). We get a standard path integral representation of the product of differential operators $e^{\pm(a_1-a_2)\partial_X}$ and $\prod_i e^{ip_i X}$ in quantum mechanics. Thus we have shown that the effective theory is consistent with description of the non-abelian degrees of freedom discussed in the first and second sections of the paper.

5. Conclusions and further directions

In this paper we have discussed the appearance of non-abelian degrees of freedom in open string theory using background independent open string field theory. The importance of the algebra of differential operators for the description of this phenomena was stressed. This algebraic structure may be helpful for the construction of the full effective action for tachyon field [21]. This action may be connected with natural "geometric" action for the algebra of differential operators analogous to the action functionals on coadjoint orbits and related symplectic manifolds considered in [41].

Obviously there is interesting mathematics behind the descriptions of the vector bundles on the submanifolds in terms of differential operators. We give a very elementary description and do not discuss the connections with D-modules, Derived categories (see for instance [42]) and its K-(co)homology invariants. The application to the universal construction of the couplings of RR gauge fields with open strings in an arbitrary D-brane background is given in [36].

Let us make at the end several remarks on the generalization of this approach to the full-fledged description of all string modes. The description in terms of the differential operators acting on the functions on the space-time obviously is an approximation of the more general structure of differential operators on the space of the half-strings (open string field theory) [28]–[32]. Consider an arbitrary open string as a composition of the "infinitesimal" open strings with identified ends. Each "infinitesimal" open string may be described by the local variables and taking into account the considerations in this paper one could suggest that the description in terms of the differential operators may be a natural candidate. Then stringy "matrix" algebra would be substituted by (subalgebra of) the universal enveloping algebra of differential operators as an abstract Lie algebra. However, the real state of the matters is more complicated. In order to deal with arbitrary functionals on the configuration space of the open strings, these "infinitesimal" strings should contain the information about all derivatives of the map of the small interval into the space-time. From the lagrangian point of view this leads to the consideration of the boundary interaction containing arbitrary derivatives of the fields (see related discussion in [21, 25]. In the hamiltonian language this is beyond the quantization of a tangent bundle of space-time. One could believe that generalization of the differential operators appearing in this more general setup would provide the right framework for the discussion of string algebra. This picture seems to be rather close to the description of the strings in Matrix Theory and deserves further considerations.

Acknowledgments

We would like to thank E. Akhmedov, M. Douglas, I. Frenkel, Hong Liu, A. Morozov, N. Nekrasov, L. Takhtajan, E. Verlinde and E. Witten. S. Sh. also would like to thank Rutgers New High Energy Theory Center for hospitality. The research of A. G. was partially supported by Grant for Support of Scientific Schools 00-15-96557 and by RFBR 00-02-16530 and the research of S. Sh. is supported by OJI award from DOE.

References

[1] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[2] K. Wilson, *Confinement of quarks*, *Phys. Rev.* **D 10** (1974) 2445.

[3] J. Kogut and L. Susskind, *Hamiltonian formulation of Wilson's lattice gauge theories*, *Phys. Rev.* **D 11** (1975) 395.

[4] A.M. Polyakov, *Gauge fields as rings of glue*, *Nucl. Phys.* **B 164** (1980) 171.

[5] A.M. Polyakov, *String theory and quark confinement*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 1 [`hep-th/9711002`].

[6] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [`hep-th/9510017`];
E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [`hep-th/9510135`].

[7] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [`hep-th/9610043`].

[8] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [`hep-th/9711200`].

[9] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [`hep-th/9802109`].

[10] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`].

[11] R. Gopakumar, S. Minwalla and A. Strominger, *Noncommutative solitons*, *J. High Energy Phys.* **05** (2000) 020 [`hep-th/003160`].

[12] K. Dasgupta, S. Mukhi and G. Rajesh, *Noncommutative tachyons*, *J. High Energy Phys.* **06** (2000) 022 [`hep-th/0005006`].

[13] J.A. Harvey, P. Kraus, F. Larsen and E.J. Martinec, *D-branes and strings as non-commutative solitons*, *J. High Energy Phys.* **07** (2000) 042 [`hep-th/0005031`];
J.A. Harvey, P. Kraus and F. Larsen, *Exact noncommutative solitons*, *J. High Energy Phys.* **12** (2000) 024 [`hep-th/0010060`].

JHEP06(2001)066

[14] D.J. Gross and N.A. Nekrasov, *Monopoles and strings in noncommutative gauge theory*, *J. High Energy Phys.* **07** (2000) 034 [hep-th/0005204];
Dynamics of strings in noncommutative gauge theory, *J. High Energy Phys.* **10** (2000) 021 [hep-th/0007204].

[15] E. Witten, *xNoncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 235.

[16] E. Witten, *On background independent open string field theory*, *Phys. Rev.* **D 46** (1992) 5467 [hep-th/9208027].

[17] E. Witten, *Some computations in background independent off-shell string theory*, *Phys. Rev.* **D 47** (1993) 3405 [hep-th/9210065].

[18] S.L. Shatashvili, *Comment on the background independent open string theory*, *Phys. Lett.* **B 311** (1993) 83 [hep-th/9303143].

[19] S.L. Shatashvili, *On the problems with background independence in string theory*, *Algebra and Anal.* **6** (1994) 215 [hep-th/9311177].

[20] S. Shatashvili, *Closed strings as solitons in background independent open string field theory*, unpublished, talk at IHES, Paris, July 1997.

[21] S.L. Shatashvili, *On field theory of open strings, tachyon condensation and closed strings*, hep-th/0105076.

[22] A. Sen, *Stable non-BPS states in string theory*, *J. High Energy Phys.* **06** (1998) 007 [hep-th/9803194]; *Stable non-BPS bound states of BPS D-branes*, *J. High Energy Phys.* **08** (1998) 010 [hep-th/9805019]; *Tachyon condensation on the brane antibrane system*, *J. High Energy Phys.* **08** (1998) 012 [hep-th/9805170];
A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, *J. High Energy Phys.* **03** (2000) 002 [hep-th/9912249].

[23] A.A. Gerasimov and S.L. Shatashvili, *On exact tachyon potential in open string field theory*, *J. High Energy Phys.* **10** (2000) 034 [hep-th/0009103].

[24] D. Kutasov, M. Marino and G. Moore, *Some exact results on tachyon condensation in string field theory*, *J. High Energy Phys.* **10** (2000) 045 [hep-th/0009148].

[25] A.A. Gerasimov and S.L. Shatashvili, *Stringy Higgs mechanism and the fate of open strings*, *J. High Energy Phys.* **01** (2001) 019 [hep-th/0011009].

[26] D.J. Gross and W. Taylor, *Split string field theory, 1*, hep-th/0105059.

[27] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, hep-th/0105058; *Boundary CFT construction of D-branes in vacuum string field theory*, hep-th/0105168.

[28] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[29] E. Witten, *Some remarks about string field theory*, *Phys. Scripta* **T15** (1987) 70.

[30] A. Strominger, *Closed strings in open string field theory*, *Phys. Rev. Lett.* **58** (1987) 629.

[31] G.T. Horowitz and A. Strominger, *Translations as inner derivations and associativity anomalies in open string field theory*, *Phys. Lett.* **B 185** (1987) 45.

[32] G.T. Horowitz, J. Lykken, R. Rohm and A. Strominger, *A purely cubic action for string field theory*, *Phys. Rev. Lett.* **57** (1986) 283.

[33] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[34] N. Nekrasov and A. Schwarz, *Instantons on noncommutative R^4 and (2,0) superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[35] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[36] E.T. Akhmedov, A.A. Gerasimov and S.L. Shatashvili, *On unification of RR couplings*, hep-th/0105228.

[37] J.A. Harvey, D. Kutasov and E.J. Martinec, *On the relevance of tachyons*, hep-th/0003101.

[38] J.A. Harvey, P. Hořava and P. Kraus, *D-sphalerons and the topology of string configuration space*, *J. High Energy Phys.* **03** (2000) 021 [hep-th/0001143].

[39] S. Merkulov, *Deformation quantization of the n-tuple point*, math.QA/9810158.

[40] A. Gerasimov, unpublished, 1991.

[41] A. Alekseev, L. Faddeev and S. Shatashvili, *Quantization of symplectic orbits of compact Lie groups by means of the functional integral*, *J. Geom. Phys.* **5** (1989) 391;
A. Alekseev and S. Shatashvili, *From geometric quantization to conformal field theory*, *Comm. Math. Phys.* **128** (1990) 197.

[42] M.R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, hep-th/0011017.